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A SPHERE ROLLING ON A HORIZONTAL ROTATING PLANE*

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It is shown that in the case of a homogeneous sphere rolling on a horizontal rotating plane the forces of viscous friction against the surrounding medium bring the center of the sphere to rest, and the resistance to rolling causes a motion along an expanding spiral. A motion of a homogeneous sphere rolling without slippage on a horizontal and sloping rotating plane was studied earlier in /1,2/, with the viscous friction and resistance to rolling both neglected. In this case the center of a sphere on a horizontal rotating plane describes a circle.

1. We consider a more general case of motion of a heavy homogeneous sphere on a rough horizontal surface Π rotating around a vertical axis at a constant angular velocity Ω. Let k be the central radius of inertia of the sphere, with the mass and radius of the latter both equal to unity. We introduce the fixed Cartesian $Oxyz$ coordinate system such that the z-axis coincides with the axis of rotation of the plane Π, and the x and y axes lie in the plane of displacement of the sphere center. Together with $Oxyz$ we also introduce a system of moving orthogonal axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in such a manner that the unit vector $\mathbf{e}_1$ is directed from the coordinate origin O towards the center of the sphere O_1 and the unit vector $\mathbf{e}_3$ along the z-axis (Fig.1). We use the polar r, γ coordinates to describe the position of the sphere center, denote by $\omega_1, \omega_2, \omega_3$ the projections of the instantaneous angular velocity of the sphere on the axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and by R_1, R_2 the projections of the reaction force on the same axes, and write the equations of dynamics under the following assumptions. The sphere moves in a viscous medium and is acted upon by the forces of resistance to rolling appearing as the viscous friction forces. Under these assumptions the equations of dynamics of the sphere have the form

$$\begin{gathered}\ddot{r} - r\dot{\gamma}^2 = R_1 - h^\circ \dot{r},\ r\ddot{\gamma} + 2\dot{r}\dot{\gamma} = R_2 - h^\circ r\dot{\gamma} \\ k^2(\dot{\omega}_1 - \omega_2\dot{\gamma}) = R_2 - h_1^\circ\omega_1,\ k^2(\dot{\omega}_2 + \omega_1\dot{\gamma}) = -R_1 - h_1^\circ\omega_2 \\ k^2\dot{\omega}_3 = -h_2\omega_3\end{gathered} \tag{1.1}$$

where h denotes the corresponding viscous friction coefficients. The condition of the sphere rolling without slippage is expressed in the form of two kinematic coupling equations

$$\dot{r} - \omega_2 = 0,\ r\dot{\gamma} + \omega_1 = r\Omega \tag{1.2}$$

Equations (1.1) and (1.2) show that the variable ω_3 is separable, i.e. the natural rotation of the sphere about the vertical axis does not affect the motion of its center.

Eliminating from (1.1) and (1.2) the quantities $\omega_1, \omega_2, R_1, R_2$ and introducing the dimensionless time $\tau = \Omega k^2(1 + k^2)^{-1}t$ and dimensionless coefficients $h = h^\circ\Omega^{-1}k^{-2}$, $h_1 = h_1^\circ\Omega^{-1}k^{-2}$, $a = 1 + k^{-2}$, we arrive at the following system of two equations:

$$\frac{\ddot{r}}{r} + (h + h_1)\frac{\dot{r}}{r} = \dot{\gamma}^2 - \dot{\gamma},\quad \frac{\dot{r}}{r} = \frac{\ddot{\gamma} + (h + h_1)\dot{\gamma} - ah_1}{1 - 2\dot{\gamma}} \tag{1.3}$$

After the substitution

$$u = \dot{\gamma},\ v = \dot{r}r^{-1},\ w = \ln r \tag{1.4}$$

the equations (1.3) reduce to a system of three first order differential equations

$$\begin{aligned}\dot{u} &= -(h + h_1)u + v - 2uv + ah_1 \\ \dot{v} &= -u - (h + h_1)v + u^2 - v^2, \\ \dot{w} &= v.\end{aligned} \tag{1.5}$$

Since the variable w has become separated, we describe the dynamics of the sphere in terms of the behavior of the representative point on the u, v phase

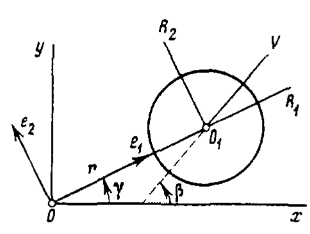

Fig.1

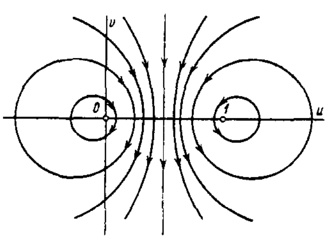

Fig.2

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.43-47,1983 plane in accordance with the differential equations

$$u^{\cdot} = -(h + h_1)\, u + v - 2uv + ah_1$$
$$v^{\cdot} = -u - (h + h_1)\, v + u^2 - v^2 \qquad (1.6)$$

The system (1.6) has two equilibrium states, (u_1, v_1) and (u_2, v_2) the character of which is determined by the roots of the characteristic equation $p_i = -(2v_i + h + h_1) \pm j\,(1 - 2u_i)$, $i = 1, 2$. When $h = h_1\,(2a - 1)$, the dynamic system (1.6) is found to be conservative, and it can be confirmed that the relation

$$u^2 + v^2 + 2ah_1 v = C\,(2u - 1) \qquad (1.7)$$

represents its integral. In this case the points (u_1, v_1) and (u_2, v_2) become center-type singularities and the family of curves (1.7) determines the decomposition of the u, v phase plane into trajectories. The decomposition coincides qualitatively with the phase pattern shown in Fig.2 for the values $h = h_1 = 0$.

It can be confirmed that when $h = h_1 = 0$, the sphere center always describes a certain circle in the fixed coordinate system (is a particular case the sphere center is at rest). Indeed, when $h = h_1 = 0$ the system of equations (1.5) has a first integral $\ln\,(u^2 + v^2)^{1/2} + w = C_1$ which can be written in terms of initial variables in the form $(r^{\cdot 2} + r^2\gamma^{\cdot 2})^{1/2} \equiv V = C_2$. Thus the sphere center moves with velocity V constant in modulo. To obtain the trajectory along which the sphere moves on the Oxy plane, we find the differential which is satisfied by the angle β (Fig.1). From $r^{\cdot} = V\cos(\beta - \gamma)$, $r\gamma^{\cdot} = V\sin(\beta - \gamma)$ we obtain $u = v\,\mathrm{tg}\,(\beta - \gamma)$. Differentiating this with respect to time and replacing $u^{\cdot}, v^{\cdot}$ by their expressions given in (1.6) we find, that at $h = h_1 = 0$, $\beta^{\cdot} = 1$. This implies that the sphere center describes, in general, a circle with constant angular velocity $\beta^{\cdot} = 1$ the value of which in real time is equal to $\omega_0 = \Omega k^2 (1 + k^2)^{-1}$ and does not depend on the initial conditions. To find how the viscous friction and resistance to rolling influence the motion of the sphere, we consider two cases.

2. A sphere on a rotating plane in a viscous medium. Here the dynamics of the sphere are determined by the behavior of the representative point on the u, v phase plane in accordance with the differential equations

$$u^{\cdot} = -hu + v - 2uv, \quad v^{\cdot} = -u - hv + u^2 - v^2 \qquad (2.1)$$

which follow from (1.6) at $h_1 = 0$. According to (2.1) we have two singularities in the u, v plane, namely $(0, 0)$ and $(1, -h)$. The first point is a stable focus and the second is an unstable focus. The isocline of the vertical tangents to the phase trajectories is a hyperbola $(1 - 2u)(2v + h) = h$, while the isocline of the horizontal tangents is a hyperbola $(1 - 2u)^2 - (2v + h)^2 = 1 - h^2$. The sign of the angle of turn of the vector field /3/ during the transition from the system of equations obtained from (2.1) at $h = 0$ to the system (2.1) is determined by the difference $\Delta = h\,(u - 1)(u^2 + v^2)$. Moreover, on the straight line $u = 1/2$ we have $u^{\cdot} = h/2 < 0$, i.e. the line $u = 1/2$ makes no contact and the limit cycle, if it exists, cannot intersect it. This implies that no limit cycles exist enveloping the singularity $(0, 0)$. We shall show that the half-plane $u > 1/2$ also contains no limit cycles enveloping the singularity $(1, -h)$. To do this we transfer the coordinate origin to this point by carrying out a variable change and setting $u = 1 + \xi$, $v = -h + \eta$. In the variable ξ, η the system (2.1) assumes the form

$$\xi^{\cdot} = -\eta\,(1 + 2\xi) + h\xi, \quad \eta^{\cdot} = \xi^2 + \xi - \eta^2 + h\eta \qquad (2.2)$$

When $h = 0$, the above system has the integral $H \equiv (\xi^2 + \eta^2)(1 + 2\xi)^{-1} = C$. Considering the family of the curves $H = C$ as a topographical Poincaré system, we compute dH/dt using the equations (2.2) to obtain $dH/dt = 2h\,(1 + \xi)(\xi^2 + \eta^2)(1 + 2\xi)^{-2}$. This implies the absence of limit cycles from the half-plane $\xi > -1/2\ (u > 1/2)$, since the sign of the expression for dH/dt does no alter when $\xi > -1/2$.

Fig.3 depicts the decomposition of the u, v phase plane into trajectories. The representative point in the u, v plane tends, irrespective of the initial conditions, to a stable singularity situated at the coordinate origin. The corresponding motion of the sphere center in the Oxy plane represents a motion along a contracting spiral towards a fixed point, the position of which is determined by the magnitude and direction of the initial velocity of the sphere center. An experiment carried out by G.G. Denisov at the Dept. of Solid Body Dynamics NII PMK of the Gor'kii University made possible the actual observation of this particular case of a sphere rolling on a rotating horizontal plane.

3. Sphere on a rotating plane in the presence of rolling resistance. In this case the dynamics of the sphere is determined by the motion of the representative point on the u, v plane in accordance with the differential equations

$$u^{\cdot} = -h_1 u + v - 2uv + ah_1, \quad v^{\cdot} = -u - h_1 v + u^2 - v^2 \tag{3.1}$$

which follow from (1.6) at $h = 0$. The isocline at the horizontal tangents to the trajectories in the u, v plane is the hyperbola

$$(2u - 1)^2 - (2v + h_1)^2 = 1 - h_1^2 \tag{3.2}$$

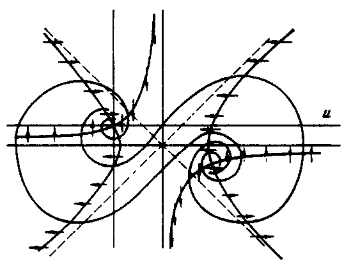

Fig.3

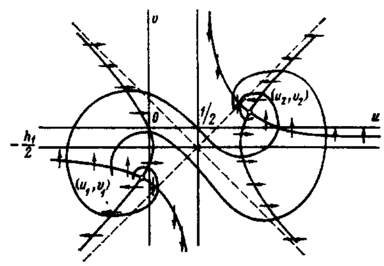

Fig.4

and the isocline of the vertical tangents in the hyperbola

$$(2u - 1)(2v + h_1) = h_1 \ (2a - 1) \tag{3.3}$$

The hyperbola (3.2) intersects the axis $v = 0$ at the point $u = 1$, and the hyperbola (3.3) at the point $u = a > 1$. The points $v = -h_1$ and $v = -ah_1 < -h_1$ represent the corresponding points of intersection of the hyperbolae with the axis $u = 0$. The points (u_1, v_1) and (u_2, v_2) of intersection of the hyperbolae (3.2) and (3.3) are singularities of the system of differential equations (3.1). The roots of the characteristic equation of (3.1) linearized in the small neighborhood of the singularities (u_1, v_1) and (u_2, v_2) are $p_i = -(2v_i + h_1) \pm j(1 - 2u_i)$, $i = 1, 2$.

According to Fig.4 $v_1 < -h_1$, therefore the singularity (u_1, v_1) is an unstable focus and (u_2, v_2) is a stable focus, since $v_2 > 0$. We shall show that for $0 < h_1 \ll 1$ at least there are no limit cycles in the u, v plane. Indeed, on the straight line $u = {}^1/_2$ we have $u^{\cdot} = h_1 (a - {}^1/_2) > 0$, everywhere, therefore if limit cycles exist, then they encircle the singularity (u_1, v_1) in the region $u < {}^1/_2$, or the singularity (u_2, v_2) in the region $u > {}^1/_2$. Consider the first possibility. We introduce the new coordinates ξ_1, η_1 with the origin at the singularity $u_1 = 0 + O(h_1^2)$, $v_1 = -ah_1 + O(h_1^2)$ and write, in these coordinates, the system of equations (3.1) as follows:

$$\xi_1^{\cdot} = (1 - 2\xi_1)\eta_1 + h_1 (2a - 1)\xi_1, \quad \eta_1^{\cdot} = \xi_1^2 - \xi_1 - \eta_1^2 + h_1 \ (2a - 1)\eta_1 \tag{3.4}$$

When $h_1 = 0$, the system has the integral $H_1 \equiv (\xi_1^2 + \eta_1^2)(1 - 2\xi_1)^{-1} = C_1$. According to the system (3.4) the time derivative $dH_1/dt = 2h_1 (2a - 1)(\xi_1^2 + \eta_1^2)(1 - \xi_1)(1 - 2\xi_1)^{-2} > 0$ for $\xi_1 < 1$. This implies the absence of limit cycles from the region $u < {}^1/_2$.

Let us consider the second possibility. We introduce new coordinates ξ_2, η_2 with the origin at the singularity $u_2 = 1 + O(h_1^2)$, $v_2 = (a - 1)h_1 + O(h_1^2)$ and write, using these coordinates, the system of equations (3.1) as follows:

$$\begin{aligned} \xi_2^{\cdot} &= -(1 + 2\xi_2)\eta_2 - h_1 (2a - 1)\xi_2 \\ \eta_2^{\cdot} &= \xi_2^2 + \xi_2 - \eta_2^2 - h_1 (2a - 1)\eta_2 \end{aligned} \tag{3.5}$$

When $h_1 = 0$, the system has the integral $H_2 \equiv (\xi_2^2 + \eta_2^2)(1 + 2\xi_2)^{-1} = C_2$. According to the system of equations (3.5) the time derivative $dH_2/dt = -2h_1 (2a - 1)(\xi_2^2 + \eta_2^2)(1 + \xi_2)(1 + 2\xi_2)^{-2} < 0$ for $\xi_2 > -1$. This implies that there are no limit cycles in the region $u > {}^1/_2$. The phase pattern depicted in Fig.4 shows that the representative point tends, on the plane u, v regardless of the initial conditions, to the stable singularity (u_2, v_2). The corresponding motion of the sphere center on the Oxy plane represents a motion along a curve approaching the expanding spiral with the center at the coordinate origin.

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